

Dynamical clustering in large populations of Rössler oscillators under the action of noise

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The effects of noise on the dynamical clustering of globally coupled chaotic Rössler oscillators are numerically investigated. Stable clusters and intermittent clustering regimes are found, depending on the coupling intensity and the noise level. Our results agree with the first experimental observations of dynamical clustering recently reported for globally coupled electrochemical oscillators.

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Since the pioneering work of Fujisaka and Yamada [1] it is known that populations of globally coupled chaotic oscillators can undergo mutual synchronization when the coupling intensity exceeds a certain threshold (see also [2]). As shown by Kaneko in his extensive numerical studies of globally coupled logistic maps (GCLM, [3]), the onset of synchronization is preceded by a wide interval of coupling intensities where dynamical clustering is observed. In this regime, some of the elements form synchronous clusters, while others may be still nonentrained. The emerging cluster partitions strongly depend on the initial conditions, so that the system has a large number of different attractors. Therefore, its statistical properties resemble those of spin glasses [3,4]. In addition to GCLM, dynamical clustering was also found for circle maps [5], Rössler oscillators [6], cross-coupled chaotic neural networks [7], and coupled biochemical reactors [8].

Dynamical clustering in globally coupled maps is robust with respect to the introduction of weak noise and quenched disorder. For instance, this behavior persists for logistic maps even when up to 20% of the connections are randomly deleted [9], or when the intensities of connections between elements are randomly modified [10]. We have also found that coupled discrete-time neural networks can undergo clustering even when a substantial fraction of cross links is eliminated [7]. Moreover, Kaneko has pointed out that the inclusion of very weak noises may improve numerical simulations of clustering in GCLM by preventing the artifacts due to digital round-off [11].

Recently, the first experimental observation of dynamical clustering was reported [12]. The investigated system represented a population of globally coupled chaotic electrochemical oscillators, where the intensity of global coupling could be varied. An essential feature of these interesting experiments was that relatively strong noise was present. To provide their detailed theoretical analysis, the influence of noise on dynamical clustering in populations of chaotic oscillators should be therefore systematically investigated. The aim of this Rapid Communication is to study the effects of weak and strong noises on dynamical clustering of chaotic Rössler oscillators. We find that this model system is able to reproduce all basic properties of dynamical clustering observed in experiments [12].

A population of N identical globally coupled chaotic Rössler oscillators under the action of independent δ -correlated noises will be considered. The model is described by the equations

$$\begin{aligned}\dot{x}_i &= -y_i - z_i + \epsilon(\bar{x} - x_i) + \xi_i(t), \\ \dot{y}_i &= x_i + ay_i + \epsilon(\bar{y} - y_i), \\ \dot{z}_i &= b - cz_i + x_iz_i + \epsilon(\bar{z} - z_i),\end{aligned}\quad (1)$$

with $i=1, \dots, N$, where ξ_i are noises with $\langle \xi_i \rangle = 0$ and $\langle \xi_i(t) \xi_j(t') \rangle = 2S \delta(t-t') \delta_{ij}$. The parameter ϵ in these equations specifies the intensity of global coupling between the oscillators, and

$$\bar{x}(t) = \frac{1}{N} \sum_{j=1}^N x_j(t), \quad (2)$$

with analogous expressions for \bar{y} and \bar{z} . Throughout this paper we take $a=b=0.2$, and $c=4.5$ so that the individual oscillators are in the chaotic regime [13].

Clustering can be identified through the examination of the distribution of instantaneous pair distances $d_{ij}(t)$ between elements, which are defined as

$$d_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}. \quad (3)$$

In the absence of noise, stable clusters can be found such that the pair distances between any two elements of the cluster asymptotically go to zero in the limit $t \rightarrow \infty$. This is, however, no longer possible when noise is included. Instead, a cluster would always represent a compact cloud of a certain size. Therefore, we define a cluster as a subset of the population such that the pair distances between any two elements in this subset are less than a certain threshold δ ; that is, $d_{ij} < \delta$ for any two elements i and j in the subset. The same definition of clustering was previously used for randomly coupled networks of logistic maps [9] and in the experiments [12]. Though the choice of δ remains arbitrary, this threshold cannot be neither too small nor too large, because in the latter case two closely lying clusters would be identified as a single

cluster. As discussed below, we have found that for the parameters indicated above a suitable choice is $\delta = 10^{-3}$.

Close to the onset of dynamical clustering (called *condensation transition* in our previous publication [6]) only a small fraction of elements is found belonging to clusters. The rest of the elements are not entrained. To measure the extent of clustering two order parameters can be introduced (see also [6,9,12]). The first of them, $r(t)$, is given by the fraction of pairs of elements (i,j) which are found at time t separated by a distance $d_{ij}(t) < \delta$. Namely,

$$r(t) = \frac{1}{N(N-1)} \sum_{i=1}^N \sum_{j=1}^N \Theta[\delta - d_{ij}(t)], \quad (4)$$

where $\Theta(x)$ is the step function, such that $\Theta(x) = 0$ for $x < 0$ and $\Theta(x) = 1$ otherwise. The second order parameter, $s(t)$, is the fraction of elements i which at time t have at least one other element j located at a distance $d_{ij} < \delta$. This can be expressed as

$$s(t) = 1 - \frac{1}{N} \sum_{i=1}^N \prod_{j=1, j \neq i}^N \Theta[d_{ij}(t) - \delta], \quad (5)$$

where the last term is simply the fraction of elements which do not have any other element within a sphere of radius δ . When complete synchronization is achieved, $r = s = 1$. If the whole population splits into clusters we have $s = 1$, whereas $r < 1$ because the distances can still be large. Finally, if some of the elements are forming clusters but other elements are still free, we find $r < s < 1$.

We have numerically solved Eqs. (1) for populations of $N = 1000$ oscillators, by means of a standard finite-difference scheme with a time increment $\Delta t = 10^{-2}$. To prepare the initial condition the system is allowed to evolve up to $t = 1000$ without coupling and noise, so that the oscillators get uniformly distributed over the Rössler attractor. Global coupling and noise are then switched on and time is reset to zero. The evolution of the coupled population is recorded up to $t = 5000$. Noise is applied at each time step of the numerical calculation by adding to the variable x_i a random number η_i , drawn from a uniform distribution in $(-\eta_0, \eta_0)$. The amplitude η_0 and the dispersion S of noise in Eqs. (1) are related by $S = \eta_0^2 / 6\Delta t$.

Noise determines a lower bound for the choice of the threshold δ . If the fluctuations introduced by noise in the size of clusters are larger than the threshold, the comparison of pair distances with δ fails to provide information on the real clustering structure of the population. In practice, for thresholds that are too low large irregular variations are observed in the long-time dependence of the order parameters r and s . For the noise amplitude considered below, the lower bound for δ is around 10^{-5} . At the other end, $\delta \geq 10^{-1}$, the order parameters are stationary for long times but show a considerable dependence on δ . For $10^{-5} < \delta < 10^{-1}$, on the other hand, r and s reach stationary values that are practically independent of δ and, thus, clustering is well defined. This justifies our choice, $\delta = 10^{-3}$.

First, we have investigated the influence of weak noise, $S = 10^{-10}$. Figure 1 shows the normalized distributions of pair distances at time $t = 5000$, for different values of the

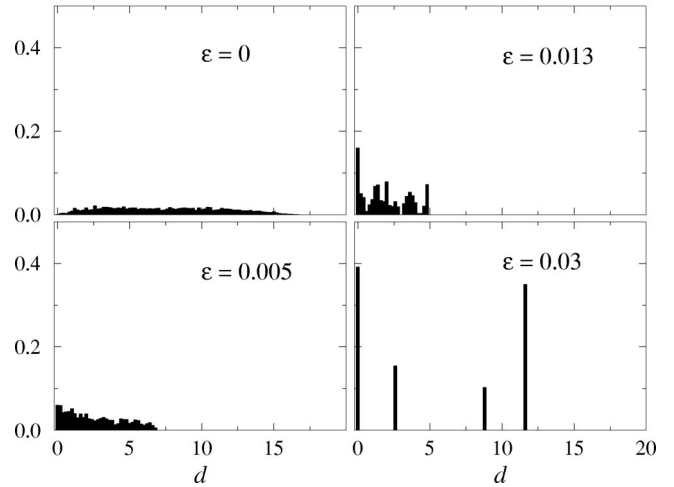


FIG. 1. Histograms of the normalized distribution of pair distances d between elements in a population of $N = 1000$ Rössler oscillators at time $t = 5000$, for various values of the coupling intensity ϵ .

coupling intensity ϵ . The distributions are plotted as 100-column histograms in the interval $0 \leq d \leq 20$. When coupling is absent ($\epsilon = 0$) the independent chaotic oscillators evolve on the Rössler attractor, and therefore pair distances are broadly distributed over a wide range. For relatively small intensities of coupling ($\epsilon = 0.005$), some changes are already found. Pair distances are substantially smaller in average, and the highest column in the histogram corresponds to the smallest distances. Further increasing the coupling intensity ($\epsilon = 0.013$), we find a strongly nonuniform distribution with some clearly distinguishable peaks. A considerable fraction of pair distances is now very close to $d = 0$. For $\epsilon = 0.03$, the histogram is formed by a few distinct lines, indicating that the population is split into a few clusters: three in this case. We stress that, as the system evolves, these histograms vary considerably. Their widths change and the relative distance between peaks successively grows and decreases as time elapses. For weak noise, however, the number of peaks does not vary after transient effects have faded out.

On the other hand, the number of clusters at long times is strongly dependent on the initial condition. In Fig. 2 we plot as dots the number of clusters found in single realizations at $t = 5000$, for many values of the coupling intensity. The line represents the mean number of clusters as a function of ϵ , obtained from the average over 20 independent realizations for each coupling intensity. A large dispersion is apparent.

We see from Fig. 2 that for very small values of the coupling intensity, $\epsilon \leq 0.004$ no clusters are detected. Then, we find a zone where the number of clusters vary irregularly, with two peaks at $\epsilon \approx 0.005$ and 0.02 . Around $\epsilon = 0.03$ there is an interval where, typically, only two to three clusters are found. For slightly larger couplings, $\epsilon \approx 0.04$, the number of clusters reaches its maximal values and shows very large dispersions. Then, from $\epsilon \approx 0.05$ on, the system is found in a regime where clustering and full synchronization, represented by a single cluster, coexist as possible attractors. Finally, for $\epsilon > 0.054$ only one cluster is observed at long times.

The same scenario can be alternatively described by

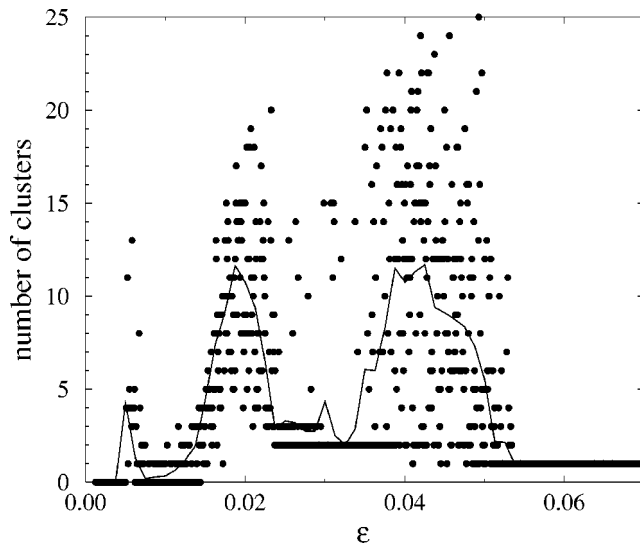


FIG. 2. Number of clusters counted at $t=5000$ for single realizations at many values of the coupling intensity ϵ (dots). The line stands for the average number of clusters over 20 independent realizations as a function of ϵ .

studying the order parameters r and s defined above as a function of the coupling intensity. Their average values over 20 realizations with different initial conditions for each value of ϵ are shown in Fig. 3. For small values of ϵ , both r and s are less than unity, indicating that not all of the oscillators are entrained in clusters. In the interval $0.016 \leq \epsilon \leq 0.032$, however, $s=1$ and $r < 1$, so that the population has fully split into clusters and no nonentrained elements are found. This situation is reverted for larger coupling intensities, around $\epsilon \approx 0.045$. Here $r, s < 1$, and thus a fraction of the population consists again of independent, nonentrained oscillators. For $0.05 < \epsilon < 0.055$ the order parameters grow steadily, revealing a rapid formation and coalescence of clusters. By $\epsilon \approx 0.058$ both order parameters have definitively reached their maximum values $r=s=1$, and full synchronization has been achieved.

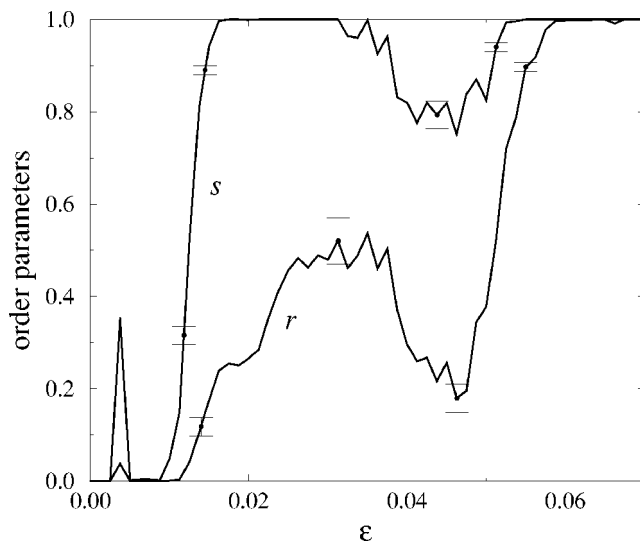


FIG. 3. Order parameters r and s as a function of the coupling intensity ϵ , averaged over 20 independent realizations. Error bars indicate the mean square deviation at selected points.

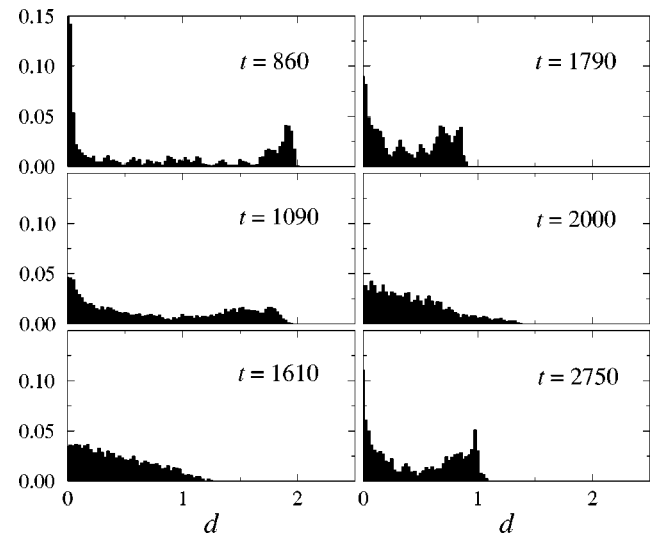


FIG. 4. Histograms over pair distances for $\epsilon=0.04$ and $S=5 \times 10^{-6}$, at different times.

So far, our numerical results correspond to the case of weak noise, where the cluster partition does not undergo significant changes with time after transients have elapsed. The situation is, however, considerably different as the intensity of noise is increased. Under the action of stronger noises, phase-space trajectories successively explore the neighborhood of the many attractors of the noiseless system [14]. Correspondingly, single elements can migrate between clusters, they can become temporarily nonentrained, and even full clusters can successively coalesce and split down as time elapses. To illustrate these effects, we have performed a series of numerical simulations at $\epsilon=0.04$ for several values of the noise strength, and using in all the realizations an initial condition that, for very weak noises, leads to an asymptotic state with two clusters. Results are particularly interesting in an intermediate interval of noise strengths, $2 \times 10^{-6} \leq S \leq 2 \times 10^{-4}$, where the population switches intermittently between a single-cluster and a two-cluster state. Figure 4 shows histograms over pair distances for a single realization with $S=5 \times 10^{-6}$, at different times. Two-cluster states are clearly defined at $t=860$ and $t=2750$, whereas a single broad cluster is observed for $t=1610$ and $t=2000$. At $t=1790$ we find a transient state with several fuzzy clusters.

The intermediate noise interval of intermittent change between one and two clusters is preceded by a regime where the two-cluster partition expected for very weak noises is preserved along the whole evolution. On the other hand, for high noises beyond the intermittency region the population is always found to form a single large cluster. The three regimes are shown in Fig. 5, where we have plotted the time evolution of histograms over pair distances as density plots, with darker tones for higher columns. For convenience in the graphical representation, pair distances are normalized at each time with respect to their maximal value $d_{\max}(t) = \max_{i,j} \{d_{ij}(t)\}$. The plots cover a time interval of 3000 time units, for three noise strengths. For weak noise [Fig. 5(a)], two sharp horizontal lines represent a persistent, well-defined two-cluster partition. For large noise [Fig. 5(c)], a wide histogram with its maximum near $d=0$ reveals a single, broad cloud. For intermediate noise strengths [Fig. 5(b)], instead,

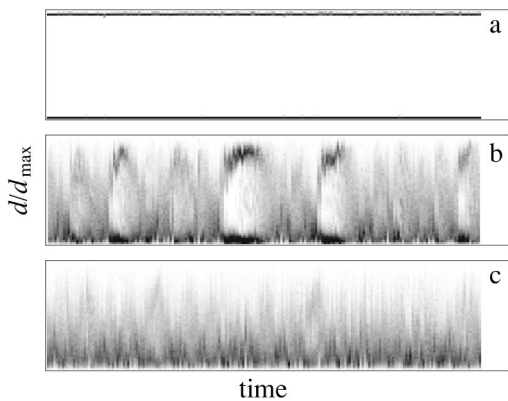


FIG. 5. Density plots of histograms over normalized pair distances as a function of time ($2000 < t < 5000$) for $\epsilon = 0.04$ and three values of the noise strength: (a) $S = 5 \times 10^{-7}$, (b) $S = 5 \times 10^{-5}$, and (c) $S = 5 \times 10^{-4}$.

the alternation of states with one and two clouds is clearly seen.

Mutual synchronization and dynamical clustering for Rössler oscillators were previously analyzed by us only in the absence of noise and for a special nonlinear form of coupling between the elements [6]. In the present study, we have used instead the standard linear form of coupling between elements [1] (also called “vector” coupling [2]). Moreover, different values of the parameters of an individual chaotic oscillator have now been chosen. Despite these differences, the observed clustering and synchronization behav-

ior is similar for the both studied systems. The introduction of weak noises does not significantly change the collective dynamics of the oscillator population, though it eliminates possible numerical artifacts caused by the computer round-off. Strong noises destroy clusters. At intermediate noise intensities, intermittent regimes have been found.

In comparing our data with the results of the experimental investigation of dynamical clustering in a population of globally coupled electrochemical oscillators [12], significant similarities can be noticed. Clustering is observed in an interval of coupling intensities preceding the transition to full chaotic synchronization. Both stable and intermittent clustering regimes are possible. When clusters are stable, the final cluster partition is strongly dependent on initial conditions and various cluster partitions can be observed at the same parameter values. Finally, the experimental dependence of the order parameter r (Fig. 11 in Ref. [12]) is similar to the respective dependence for the Rössler oscillators (Fig. 3). These similarities are remarkable, especially if we take into account that the dynamics of individual chaotic oscillators and coupling between them were different for the experimental system and the studied mathematical model. They suggest that dynamical clustering is a robust universal phenomenon which can be expected in various large dynamical systems formed by globally interacting elements with chaotic individual dynamics.

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